



Göteborg ITP preprint
hep-th/0410110
October, 2004

Thoughts on Membranes, Matrices and Non-Commutativity

Invited talk presented at the conference “Non-Commutative Geometry and Representation Theory in Mathematical Physics”, Karlstad, Sweden, July 5-10, 2004

Martin Cederwall

*Department of Theoretical Physics
Göteborg University and Chalmers University of Technology
S-412 96 Göteborg, Sweden*

Abstract: We review the passage from the supermembrane to matrix theory via a consistent truncation following a non-commutative deformation in light-cone gauge. Some indications are given that there should exist a generalisation of non-commutativity involving a three-index theta on the membrane, and we discuss some possible ways of investigating the corresponding algebraic structure.

1. Introduction

String theory, or M-theory, which seems a more proper name in the present context, possesses vacuum solutions in which there are no perturbative excitations, and (thus) no strings. Among these is the 11-dimensional Minkowski space (among other solutions to 11-dimensional supergravity), in which the effective low-energy excitations are described by 11-dimensional supergravity. In this background, M-theory allows the existence of certain extended objects, namely the M2- and M5-branes, which can be seen *e.g.* by considering solutions to the supergravity equations of motion. The M2-brane, or 11-dimensional supermembrane, is electrically charged with respect to the 3-form tensor field of the supergravity theory, and the M5-brane is magnetically charged (in addition there are objects that carry gravitational charges, but these will not be considered here). In the absence of strings, one may ask if membranes may play the rôle of fundamental excitations of M-theory, albeit non-perturbative. Such a speculation is supported by the nature of the corresponding supergravity solution, which has a singularity (the M5-brane is completely smooth, which is typical for a truly solitonic object). It shows a difference, however, compared to fundamental string solutions in 10 dimensions exhibiting naked singularities in the string frame, in that the singularity is surrounded by a horizon, probably indicating the non-perturbative nature of membranes. There does not seem to exist a limit where membrane states are perturbative.

Quantisation of membranes [1] has been a challenging problem since their theoretical discovery (for reviews of membranes and matrix theory, see ref. [2]), and there is at this point no complete solution of the problem. It is generally believed that a solution should provide valuable insight into the non-perturbative nature of string theory/M-theory, although presumably not in a background-independent form. Note that since there is no perturbation theory, there is no such distinction as between a first-quantised and a second-quantised theory, and membrane quantisation has to be very different from string quantisation (producing the one-string states used to build a multi-string Fock space).

Most of the progress in this area of M-theory has been made in terms of matrix theory. Matrix theory is in a well-defined way an approximation, or even a consistent truncation, of membrane theory. The first section of this talk will be devoted to a demonstration of the relation between membranes and matrices. In this procedure, non-commutativity plays a crucial (mathematical) rôle. In the subsequent sections, we will discuss the physical relevance of this non-commutativity, and also comment about its generalisations and other instances where non-commutativity plays a rôle in M-theory.

2. From Membranes to Matrices

The action of an M2-brane moving in a background which is a solution to the 11-dimensional supergravity equations of motion is [3]

$$S = -T \int_{\mathcal{M}_3} d^3\xi \sqrt{-g} + T \int_{\mathcal{M}_3} C . \quad (2.1)$$

Here, \mathcal{M}_3 is the membrane world-volume with coordinates $\{\xi^i, i = 0, 1, 2\}$ and T is the membrane tension, proportional to ℓ_P^{-3} , ℓ_P being the 11-dimensional Planck length. The fields g and C occurring in the action are pullbacks to the membrane world-volume of the corresponding background superfields, which means that the superspace coordinates X^m , $m = 0, \dots, 11$ and θ^μ , $\mu = 1, \dots, 32$ are dynamical variables for the supermembrane. In the following, we will, for sake of simplicity, exhibit only the bosonic variables, although we will keep in mind that we are working with the supersymmetric membrane, and comment at relevant places on implications of supersymmetry. One crucial aspect of the supermembrane (as of any half-BPS object) is κ -symmetry, implying that the action (2.1) is invariant under local translations in 16 of the 32 fermionic directions, so the number of physical fermionic variables is 16. On-shell, we thus have 8 fermions and 8 bosons (the transverse fluctuations, the longitudinal ones are killed by diffeomorphisms on \mathcal{M}_3). We will also in flat 11-dimensional Minkowski space with vanishing C -field. The C -field itself is relevant to another instance of “non-commutativity”; this will be commented upon later.

In general, we should consider all possible topologies (*i.e.*, 2-dimensional topologies at a given time) of the membrane, and keep in mind the possibility of dynamical topology change. However, topology seems to be quite unimportant in membrane theory. Here, we will only give a heuristic argument for this. Consider a membrane consisting of two surfaces connected by a very thin tube. The action for such a configuration will in the limit where the tube becomes infinitesimally thin only differ infinitesimally from the action for the configuration where there are two membranes, meaning that (classical) membranes may grow “spikes”, connecting different parts or going towards infinity, at no cost of energy. It also seems to indicate that a configuration with one membrane already contains configurations with two or more membranes. As we will soon see, this property has to do with the existence of flat directions in the potential. It is of course important to check whether this property persists at the quantum level. It has been proved that it does not for a bosonic membrane, due to zero-point fluctuations in directions transverse to the flat directions, so that quantum states of a bosonic membrane are localised. In a supersymmetric situation however, the flat directions persist, due to cancellations of zero-point energies between bosons and fermions. This gives an indication that topology should be irrelevant for supermembranes, and that

“first-quantised” supermembrane Hilbert space is large enough to contain an entire multi-particle Fock space. In this respect, membranes behave very differently from strings. Strings have no flat directions—a string stretched between two strings costs energy, and topology change (change in the number of disconnected components) represents interaction. The continuity of the supermembrane spectrum, reflecting its multi-particle nature, was at first taken as a signal that membrane quantisation produced undesired (for a first-quantised theory) physical results [4]. For these reasons, and for sake of simplicity, we will consider membranes whose topology is a torus T^2 .

The diffeomorphism constraints for a membrane are

$$\begin{aligned}\mathcal{T}_i &\equiv P \cdot \partial_i X \approx 0 , \\ \mathcal{H} &\equiv \frac{1}{2T} P^2 + \frac{T}{2} \tilde{g} \approx 0 ,\end{aligned}\tag{2.2}$$

$\tilde{g}_{ij} = \partial_i X^a \partial_j X^b \eta_{ab}$ being the induced metric on the two-dimensional membrane space-sheet, and \tilde{g} its determinant (from here on, $i, j, \dots = 1, 2$ are tangent indices on the space-sheet). It is conveniently rewritten in terms of the “Poisson bracket” on the space-sheet,

$$\{A, B\} = \varepsilon^{ij} \partial_i A \partial_j B ,\tag{2.3}$$

as $\tilde{g} = \frac{1}{2} \{X_a, X_b\} \{X^a, X^b\}$, so that the hamiltonian constraint takes the form

$$\mathcal{H} = \frac{1}{2T} P^2 + \frac{T}{4} \{X_a, X_b\} \{X^a, X^b\} \approx 0 .\tag{2.4}$$

We now want to go to a light-cone gauge. This is achieved by aligning world-volume time with a light-like direction in target space, $X^+ = x^+ + p^+ \tau$, while the same component of the momentum is demanded to be constant*, $P^+ = p^+$. These gauge choices, together with the constraints, allow for solution of P^- and X^- , except for the constant mode of the latter (here, and in the following, we set $T = 1$):

$$\begin{aligned}P^- &= \frac{1}{2p^+} (P^I P^I + \frac{1}{2} \{X^I, X^J\} \{X^I, X^J\}) , \\ \partial_i X^- &= \frac{1}{p^+} P^I \partial_i X^I ,\end{aligned}\tag{2.5}$$

* In this is many other equations, one should insert an auxiliary density on the membrane space-sheet, since the momentum, for example, is a density. On the torus, we can choose it to be a constant.

where the index I now runs over the transverse light-cone directions $I = 1, \dots, 9$. P^- is the light-cone hamiltonian. The physical phase space is spanned by X^I , P^I , whose Dirac brackets, as usual in light-cone gauge, are identical to the original Poisson brackets[†], $[X^I(\xi), P^J(\xi')]_{PB} = \delta^{IJ} \delta^{(2)}(\xi - \xi')$. From the form of $\partial_i X^-$ in eq. (2.5) it follows as an integrability condition that $J \equiv \{X^I, P^I\} \approx 0$. This is a remaining gauge invariance, the area-preserving diffeomorphisms (APD's) of the membrane space-sheet [5,6].

If we expand the fields in momentum eigenfunctions on T^2 labeled by a pair of integers k_i , the Poisson bracket between two functions is

$$\{e^{ik \cdot \xi}, e^{ik' \cdot \xi}\} = -\epsilon^{ij} k_i k_j e^{i(k+k') \cdot \xi} . \quad (2.6)$$

This is also the algebra (under $[\cdot, \cdot]_{PB}$) between modes of the generator J of APD's. It is well-known that this algebra in a certain sense is isomorphic to “ $u(\infty)$ ” [7]. Namely, a parametrisation of $u(N)$ generators in terms of “clock and shift matrices” gives the $u(N)$ commutation relations as

$$[T_k, T_{k'}] = -2i \sin\left(\frac{\pi}{N} \epsilon^{ij} k_i k_j\right) T_{k+k'} , \quad (2.7)$$

for pairs of integers k, k' ranging from 0 to N . Clearly, eq. (2.7) provides a “regularisation” of eq. (2.6), in the sense that, for large N , the modes with mode numbers $\ll \sqrt{N}$ (low enough to approximate the sine function with a linear function) behave the same way (modulo rescaling) in both, while the number of generators in eq. (2.7) is finite^{*}.

It seems therefore physically motivated to regulate the algebra of functions and of the APD's by replacing it by $u(N)$. Here we want to advocate the view, however, that the best way of reaching that result is by *consistent truncation*. A consistent truncation of a dynamical system consists of a choice of a subset of modes, whose excitations do not force the excitations of the remaining modes, *i.e.*, setting the remaining modes to zero is consistent with the equations of motion of the system. This is clearly not achieved by approximating

[†] This is the ordinary dynamical Poisson bracket, not to be confused with the structure introduced in eq. (2.3).

^{*} From the literature, one may get the impression that there is some disagreement on the relation between the algebra of APD's and $u(\infty)$ (see *e.g.* ref. [8]). This confusion can be attributed to observations that if one keeps *all* generators in eq. (2.7), *i.e.*, also those beyond the scope of the linear approximation, there is no way to obtain eq. (2.6) as an $N \rightarrow \infty$ limit. In a physical interpretation, one has to invoke a kind of low-energy approximation, the correct statement being that, for any *fixed* k 's, the limit of the $su(N)$ commutators is given by eq. (2.6). Also, depending on topology, there may be APD's not connected to the identity. With regard to the discussion on topology, we believe it to be physically motivated not to consider these.

the algebra as above. The procedure we advocate is instead, as stressed *e.g.* in ref. [9], a non-commutative deformation of the algebra of functions, followed by a consistent truncation. In that way, one will regain full control over any algebraic property of the theory, like *e.g.* supersymmetry.

The algebra of functions is deformed with a non-commutativity parameter θ by $[\xi^1 = \sigma, \xi^2 = \varrho] = i\theta$ and Weyl ordering, so that

$$f \star g = f e^{\frac{i}{2} \theta \varepsilon^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g . \quad (2.8)$$

The Fourier modes of definite momentum then commute as

$$[e^{ik_i \xi^i}, e^{ik'_i \xi^i}]_\star = -2i \sin(\frac{1}{2} \theta \varepsilon^{ij} k_i k'_j) e^{i(k_i + k'_i) \xi^i} , \quad (2.9)$$

If θ is “rational”, $\theta = \frac{2\pi q}{N}$ where q and N are coprime integers, the sine function in the structure constants will have zeroes. The functions $e^{isN\sigma}$ and $e^{irN\varrho}$ ($s, r \in \mathbb{Z}$) commute with all other functions, they are central elements in the algebra. This means that they can be consistently modded out from the algebra of function under the star product, since left and right multiplication coincide on all functions. If the star products by $e^{iN\sigma}$ and $e^{iN\varrho}$ are identified with the identity operator, one obtains the equivalences

$$\begin{aligned} e^{i(k+N)\sigma + il\varrho} &\approx (-1)^l e^{ik\sigma + il\varrho} , \\ e^{ik\sigma + i(l+N)\varrho} &\approx (-1)^k e^{ik\sigma + il\varrho} . \end{aligned} \quad (2.10)$$

The star-commutator algebra after this “consistent truncation” is identical to eq. (2.7), $u(N)$.

To summarise, matrix theory is obtained from membrane theory in two steps, the first one being a non-commutative deformation with $\{\cdot, \cdot\} \rightarrow [\cdot, \cdot]_\star$, the second one a consistent truncation with $[\cdot, \cdot]_\star \rightarrow [\cdot, \cdot]_{N \times N}$. A less strict and more intuitive way of explaining the need for a deformation in order to enable a consistent truncation would be that the (rational) deformation effectively introduces a discretisation of the torus, in which the APD’s act by finite translations, while they in the undeformed case act by infinitesimal ones. In this way the deformed algebra will contain elements that translate all the way around a periodic directions, while this does not happen in the undeformed case, accounting for the impossibility of finding an $su(N)$ subalgebra of “ $su(\infty)$ ”.

Modulo constants, the light-cone hamiltonian for the matrix theory is

$$P^- \sim \frac{1}{2p^+} \text{Tr} \left(P^I P^I + \frac{1}{2} [X^I, X^J] [X^I, X^J] \right) + \text{terms with fermions} , \quad (2.11)$$

where the X 's and P 's are $N \times N$ matrices, which is proportional to the hamiltonian for the dimensional reduction to $D = 1$ of $D = 10$ SYM with gauge group $U(N)$.

The flat directions of the potential in eq. (2.11) form the “classical moduli space”. This is easily seen by diagonalisation to be $\mathbb{R}^9 \times (\mathbb{R}^{9(N-1)}/P_N)$, P_N being the permutation group of N objects. So, while the diagonal matrices can be seen as parametrising the motion of N “partons”, the off-diagonal entries represent interactions. It should be stressed that this multi-particle interpretation of the one-membrane Hilbert space provides an interesting alternative, in terms of ordinary quantum mechanics, to quantum field theory. For the supersymmetric membrane, as mentioned earlier, the existence of the flat directions persists at the quantum level. The full non-linear problem seems very difficult, so far only the form of the asymptotic scattering states (far out in the valleys) has been found exactly, along with low-order series expansions in velocities for interactions [10]. Still, index theorems ensure that a unique ground state exists [11,12]. This is crucial for the connection to M-theory; then the fermionic zero-modes generate the 2^8 physical fields of 11-dimensional supergravity[†].

The corresponding procedure with deformation and consistent truncation is applicable in other situations, *e.g.* when the membrane winds a compactified circle, in which case matrix string theory is produced [13,9], or in backgrounds with C -field, which gives rise to a deformation of the SYM theory [14].

3. The C -Field and Non-Commutativity

In string theory, field theories on D-branes, in the presence of a background 2-form B -field, can be seen as non-commutative. The non-commutativity parameter is directly related to the B -field. One way of deriving this is to consider the mixed Dirichlet/Neumann boundary condition for a string ending on the D-brane as a set of constraints and deriving the corresponding Dirac brackets for the string end-point coordinates. In 11 dimensions, membranes may end on M5-branes in an analogous way, and one would expect a similar mechanism to be at work. The situation is complicated by the limited knowledge of the theory on M5-branes relevant for the situation. A simplistic approach using the membrane action yields a non-commutative loop space with a three-index theta [15]. The precise form of the

[†] The 16 fermionic zero-modes act as gamma matrices of $so(16)$. Under the “Hopf reduction” $so(16) \rightarrow so(9)$, the fermionic Hilbert space, an $so(16)$ spinor, decomposes as $128_s \rightarrow 44 \oplus 84$ (the metric and C -field) and $128_c \rightarrow 128$ (the gravitino).

non-commutativity is not known, due to the non-linearity of membrane dynamics. Similar results, with three-index thetas, have been obtained from supergravity solutions [16,17,18]. The precise meaning is so far much more elusive than in string theory.

4. From Two-Index to Three-Index?

Both instances of non-commutativity in membrane theory, and their relation to string theory, can be taken as indicating that an algebraic structure should be lifted/generalised to some structure involving three indices. In the case of the previous section, the string theory B -field comes from a C -field in 11 dimensions. In the case of non-commutativity on the membrane, as described in section (2), θ is introduced as a mathematical tool, and unless it has a more profound physical meaning (see the following section), one might regard it as specific to light-cone gauge. It is nevertheless tempting to speculate in “covariant” three-index structures involving the entire world-volume, not only the space-sheet.

Some authors have argued that a formulation in terms of Nambu brackets should be relevant for membrane dynamics [19,20,21]. The Nambu bracket is the straight-forward generalisation of the Poisson bracket to a “three-product”, $\{A, B, C\} = \epsilon^{ijk} \partial_i A \partial_j B \partial_k C$. In addition to being completely antisymmetric, it obeys a derivation property (like the Poisson bracket) and a “Fundamental Identity” (FI, generalising the Jacobi identity for Poisson brackets):

$$\begin{aligned} \{AB, C, D\} &= A\{B, C, D\} + \{A, C, D\}B, \\ \{A, B, \{C, D, E\}\} &= \{\{A, B, C\}, D, E\} + \{C, \{A, B, D\}, E\} \\ &\quad + \{C, D, \{A, B, E\}\}. \end{aligned} \tag{4.1}$$

If one wishes to deform the Nambu bracket, abstract algebra may not give enough information on how to do this. One should think about the physical meaning of *e.g.* the FI, in order to know whether it should hold in the deformed case. Of course, if the Nambu bracket $\{A, B, X\}$ is interpreted as a transformation with parameters A and B on X (three-dimensional volume-preserving diffeomorphisms), the FI is necessary for the closure of the transformations to a Lie algebra. On the other hand, although the membrane action may be rewritten in terms of Nambu brackets, the symmetries of the membrane are full diffeomorphisms and not expressible in terms of Nambu brackets, which makes the situation somewhat unclear. An attempt to deform the Nambu bracket with a three-index θ fails, if one tries a naïve expansion in momenta (the same kind of deformation that deforms the Poisson bracket to a star commutator). Other methods have been devised, see *e.g.* ref. [22], which relies on factorisation of polynomials.

Our opinion is that in order to get further on this track, one would need some intuition concerning the physical meaning of a three-structure. A relaxation of some of the identities [20] would need to be motivated and replaced by some weaker identity, in order not

to end up in a situation where, for example, it is no longer possible to realise “symmetries” infinitesimally as operators on a Hilbert space (unless one has clear reasons for this).

An inventory of other mathematical “three-structures” include *e.g.* Jordan triple products (with a defining identity which, in spite of different symmetry properties, can be written in a form identical to the FI of the Nambu bracket) and the 3-algebras of ref. [23]. It is not clear why either of these structures should be relevant to membranes, but at a purely mathematical level there are connections between these and the brackets of ref. [20].

5. Symmetries of Non-Commutative Membranes?

The membrane “partons” lack some properties to make them independent particles in 11 dimension. Even if they move independently in the 9 transverse directions, there is only one collective momentum p^+ . On reduction to $D = 10$ matrix theory [24], and type IIA, so called “discrete light-cone quantisation” (DLCQ, which will not be discussed in detail here, see *e.g.* ref. [2]), they become D0-branes. In that process the size N of the matrices is proportional to p^+ . We now come to some serious questions concerning the physical interpretation of the truncated theory in 11 dimensions. Even if we have kept as much as possible of the algebraic structure of the theory, so that we are ensured that any linearly realised symmetry of the original theory will pertain, some symmetries are non-linearly realised in light-cone gauge. This happens especially for the dynamical generators p^- and j^{-I} of the Poincaré algebra and half of the supersymmetry generators.

Several authors have investigated the Poincaré invariance of the truncated supermembrane, with the conclusion that it is broken by the truncation, but recovered in the $N \rightarrow \infty$ limit. This should not be surprising: it does not seem reasonable to expect such a symmetry in an interacting theory when particle number is restricted (remember that the dynamical generators are exactly the ones that transform out of the initial value surface). The identification of p^+ with particle number in the DLCQ procedure also indicates this. The relation between N and the non-commutativity parameter θ then seems to indicate that if there is a covariant version of matrix theory, one should allow θ to transform. This would be possible if it happens after the deformation, but before the truncation. We think that another advantage of the two-step consistent truncation procedure may be that it allows this kind of line of thought.

One could discuss at length whether such an attempt would be successful. For example, θ is a scalar from target space point of view, so its introduction might still be consistent with Poincaré symmetry. Also, from a world-volume perspective, introduction of non-commutativity does not break rotational symmetry, but deforms it (see the talk by P. Kulish at this conference [25]). Probably, only a direct construction of the dynamical generators, or a proof of the impossibility thereof will settle the question. Work in this direction

is under way. Here we would only like to mention that the replacement of APD's by gauge transformations may give a clue. Namely, if we still expect the remaining gauge symmetry to arise as an integrability condition on X^- , we are lead to another expression for X^- involving a superposition of different values of θ ,

$$\partial_i X^- = \frac{1}{2p+\theta} \int_0^\theta d\theta' (\partial_i X^I \star_{\theta'} P^I + P^I \star_{\theta'} \partial_i X^I) . \quad (5.1)$$

which is easily verified by noting that differentiating with respect to θ brings down two derivatives. Whether insertion of this new relation, together with other suitable modifications, may lead to Lorentz covariance remains to be seen.

Acknowledgements: The author wants to thank the organisers of the conference for a stimulating experience and Jens Hoppe and Daniel Sternheimer for comments and information concerning their work.

REFERENCES

- [1] B. de Wit, J. Hoppe and H. Nicolai, “*On the quantum mechanics of supermembranes*”, Nucl. Phys. **B305** (1988) 545.
- [2] M.J. Duff, “*Supermembranes*”, hep-th/9611203;
H. Nicolai and R. Helling, “*Supermembranes and (M)atrix theory*”, hep-th/9809103;
B. de Wit, “*Supermembranes and super matrix models*”, hep-th/9902051;
W. Taylor, “*(M)atrix theory: Matrix quantum mechanics as fundamental theory*”,
hep-th/0101126;
T. Banks, “*TASI lectures on matrix theory*”, hep-th/9911068.
- [3] E. Bergshoeff, E. Sezgin and P.K. Townsend, “*Supermembranes and eleven-dimensional supergravity*”, Phys. Lett. **B189** (1987) 75; “*Properties of the eleven-dimensional super membrane theory*”, Ann. Phys. **185** (1988) 330.
- [4] B. de Wit, M Lüscher and H. Nicolai, “*The supermembrane is unstable*”, Nucl. Phys. **B320** (1989) 135.
- [5] E. Bergshoeff, E. Sezgin, Y. Tanii and P.K. Townsend, “*Super p-branes as gauge theories of volume preserving diffeomorphisms*”, Ann. Phys. **199** (1990) 340.
- [6] B. de Wit, U. Marquard and H. Nicolai, “*Area preserving diffeomorphisms and supermembrane Lorentz invariance*”, Commun. Math. Phys. **128** (1990) 39.
- [7] D.B. Fairlie and C.K. Zachos, “*Infinite dimensional algebras, sine brackets and $su(\infty)$* ”, Phys. Lett. **B224** (1989) 101.
- [8] J. Swain, “*On the limiting procedure by which $SDiff(T^2)$ and $su(\infty)$ are related*”, hep-th/0405002.
- [9] M. Cederwall, “*Open and winding membranes, affine matrix theory and matrix string theory*”, J. High Energy Phys. **0212** (2002) 005 [hep-th/0210152].
- [10] J. Hoppe, “*Zero energy states in supersymmetric matrix models*”, Class. Quantum Grav. **17** (2000) 1101, and references therein.

- [11] S. Sethi and M. Stern, “*D-brane bound states redux*”, Commun. Math. Phys. **194** (1998) 675 [hep-th/9705046].
- [12] V.G. Kac and A.V. Smilga, “*Normalized vacuum states in $N=4$ supersymmetric Yang–Mills quantum mechanics with any gauge group*”, Nucl. Phys. **B571** (2000) 515 [hep-th/9908096].
- [13] Y. Sekino and T. Yoneya, “*From supermembrane to matrix string*”, Nucl. Phys. **B619** (2001) 22 [hep-th/0108176].
- [14] M. Cederwall and H. Larsson, “*M5-branes and matrix theory*”, in the proceedings of the International Seminar on Supersymmetries and Quantum Symmetries (SQS '03), Dubna, July 2003 and of the 9th Adriatic Meeting, Dubrovnik, Sept. 2003, hep-th/0312303.
- [15] E. Bergshoeff, D.S. Berman, J.P. van der Schaar and P. Sundell, “*A non-commutative M-theory five-brane*”, Nucl. Phys. **B590** (2000) 173 [hep-th/0005026].
- [16] D.S. Berman, M. Cederwall, U. Gran, H. Larsson, M. Nielsen, B.E.W. Nilsson and P. Sundell, “*Deformation-independent open brane metrics and generalised theta parameters*”, J. High Energy Phys. **0202** (2002) 012 [hep-th/0109107].
- [17] J.P. van der Schaar, “*The reduced open membrane metric*”, J. High Energy Phys. **0108** (2001) 048 [hep-th/0106046].
- [18] D.S. Berman and B. Pioline, “*Open membranes, ribbons and deformed Schild strings*”, Phys. Rev. **D70** (2004) 045007 [hep-th/0404049].
- [19] J. Hoppe, “*On M algebras, the quantisation of Nambu mechanics, and volume preserving diffeomorphisms*”, Helv. Phys. Acta **70** (1997) 302 [hep-th/9602020].
- [20] H. Awata, M. Li, Dj. Minić and T. Yoneya, “*On the quantization of Nambu brackets*”, J. High Energy Phys. **0102** (2001) 013 [hep-th/9906248].
- [21] C. Zachos, “*Membranes and consistent quantization of Nambu dynamics*”, Phys. Lett. **B570** (2003) 82 [hep-th/0306222].
- [22] G. Dito, M. Flato, D. Sternheimer and L. Takhtajan, “*Deformation quantization and Nambu mechanics*”, Commun. Math. Phys. **183** (1997) 1 [hep-th/9602016].
- [23] R.J. Lawrence, “*On algebras and triangle relations*”, in proceedings of “Topological and geometrical methods in field theory”, Turkku, May 1991.
- [24] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, “*M theory as a matrix model: a conjecture*”, Phys. Rev. **D55** (1997) 5112 [hep-th/9610043].
- [25] M. Chaichian, P.P. Kulish, K. Nishijima and A. Tureanu, “*On a Lorentz-invariant interpretation of non-commutative space-time and its implications on non-commutative QFT*”, hep-th/0408069.